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On the specification of models of optimal production planning under risk

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ON THE SPECIFICATION OF MODELS OF OPTIMAL PRODUCTION PLANNING UNDER RISK

by

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1. GENERAL INTRODUCTION

The concept of risk has played a very significant role in the theory of production, resource allocation, and the theory of statistical decisions. It is not intended here to include in any comprehensive manner the various economic and statistical aspects of the concept of risk in economic theory. Only a limited objective of analyzing the implications of a certain type of probabilistic concept of risk in models of planning for optimal production in a linear programming framework has been attempted. The plan of the discussion covering this topic will be as follows:

In Chapter 1 is presented a general introduction to economic models involving risk and uncertainty at different phases and restrictions, and constraints in the form of either equalities or inequalities. This is followed by Chapter 2, which presents a brief survey of some of the most important operational results available in the theory of linear programming. These operational results have variations in some or all of the parameters such as prices, input coefficients, and resources. The concept of risk implied by these variations is taken purely in the statistical sense in terms of probability measures. The distinction between prior and posterior statistical distribution is not made. Chapter 3 examines the analytical methods of sensitivity analysis and the parametric approach applied to linear programming models with variation in the parameters. The concept of risk implied in these variations is not statistical and the emphasis here is on deriving simple decision rules for guaranteeing a degree of safety. In Chapter 4 an empirical model of probabilistic planning applied to the production situation of a farm in the Chincha Valley of Peru is presented. The model used is a variant of a simple linear programming model with eleven crops and four types of quarterly restrictions: (1) water availability, (2) land, (3) capital and (4) labor. The variations in the net incomes are the net prices in the objective function caused by the variability of yield and prices received by the farmers. The empirical example is considered only for illustrative purposes since it provides a very simple case of production planning under risky conditions. Hence, the lines of generalizing the empirical results, which have limitations due to the data situation and special conditions of the particular geographic region of Peru, have been indicated in appropriate places in Chapter 4.

Finally, a broad summary of all principal results is presented in Chapter 5.

1.1. Risk in Economic Theory, Models with Equalities

Many hypotheses about individual or corporate economic behavior, under uncertainty and risk, attempt to deal with the problem of behavior under the assumption that it is reasonable for the behavioral unit to maximize gain or profit.

The difference between uncertainty and risk must be pointed out here. Each term has had distinct meanings in different parts of economic literature. The term "risk" is characterized in a model in which the entire probability distribution of the outcomes has formally been taken into account, whether the character of that distribution is considered subjective or objective. The term "uncertainty" is applied to models in which the above stated conditions are not the case.

It is very important to have a realistic theory explaining how individuals choose among alternate courses of action when the consequences of their actions are not fully known to them. A survey of the literature of approaches to the theory of choices in risk-taking situations has been given by Arrow (3).

The probability theory represents the sustained efforts of mathematicians and philosophers to provide a rational basis on which expectations may be derived from past events. Roy (33) stated that there are major objections when one attempts to maximize expected gain or profit. The ordinary man has to consider the possible outcomes of a given course of action on one occasion only, and the average or expected outcome, if this conduct were repeated a large number of times under similar conditions, is irrelevant. Also, the well-known phenomemon of the diversification of resources among a wide range of project or investment situations is not explained.

Since the origin of the species, men have been making decisions, and other men have been telling them how they either make, or should make, decisions. von Neumann and Morgenstern (30) developed a theory of maximizing the expected utility. In order for their results to be valid, however, their assumption that rational individuals are choosing the right utility functions must hold true.

The fundamental problem of production is the optimum allocation of scarce resources between alternative ways of achieving an objective. It can be seen that the objective may be the maximization of the firm's profits or the minimization of costs. Cases exist, however, in which besides profit maximization or cost minimization, the objective includes risk minimization. If the decision-maker is willing to sacrifice profit in exchange for security, the result depends on his behavior.

The firm is engaged in a type of "game against nature," an opponent which is really not a malevolent, maximizing rival acting purposely to thwart the firm's designs. Yet, one possible approach to the firm's decision-making is to assume it acts as if the intention for optimal solutions were that for a game in which Nature did indeed have those attributes.

A list of the best-known criteria may be found in the work of Van Moeseke (27).

Expected profit is an appropriate maximum in recurrent "small" decisions, but where disaster is possible one may prefer reduced profit with less risk. In terms of the probability distribution of the relevant outcome variables, the question is whether to consider the mean only or also the variance or other measures of dispersion or skewness.

Markowitz (26) has applied concepts of programming under uncertainty to selection of investment portfolios. Assume that one unit of money is to be subdivided into amounts $x_1, \ldots x_n$ for the purchase of corresponding amounts of n assets. Then $\sum_i x_i = 1$. Assume known the joint probability distribution $F(r_1, \ldots, r_n)$ of return on assets $1, \ldots, n$, with means,

$$\mu_i = \int r_i \, dF$$

and covariance

$$\sigma_{ij} = \int (r_i - \mu_i)(r_j - \mu_j) \, dF$$

The problem is to choose what can be called an efficiency portfolio in accordance with the a priori probability distribution F.

The traditional rule used in economic theory has been to discount the expected return μ_i for each asset by some formula that takes account of its degree of risk as measured by σ_{ii}, and maximize total discounted expected return. In fact, this rule is hardly ever used in practice; the overwhelming practice is to diversify holdings, whereas the rule leads in general to the selection of one single preferred asset.

Markowitz defines the efficiency of a portfolio $(x_1, \ldots x_n)$ in terms of the relation to its expected return,

$$\mu = \sum_i \mu_i x_i$$

and its variance of return,

$$\sigma^2 = \sum_i \sigma_{ij} x_i x_j$$

to the expected return $\bar{\mu}$ and variance $\bar{\sigma}^2$ of alternative portfolios of the same purchase price.

$$\sum_i x_i = \sum_i \bar{x}_i$$

The portfolio $(x_1, \ldots x_n)$ is called efficient if there exists no such alternative portfolio with

$$\bar{\mu} \geq \mu \qquad \bar{\sigma}^2 \leq \sigma^2$$

except possibly with both equality signs holding.

In the classical techniques for applying calculus to certain types of optimization problems, it is possible to use the classical theory to solve analytically for an optimal solution in terms of the various parameters appearing in the problem.

The following example is given to illustrate the above-mentioned possibility.[1] Consider a machine part which is produced on a particular lathe in a machine shop. The diameters of the parts turned out will not always be precisely the same, but will vary somewhat from one piece to another due to a variety of causes. The diameter x of any particular piece can be considered to be a random variable. The mean μ of this random variable can be varied appropriately modifying the lathe setting. It is given a density function for x as $f(x;\mu)$. In order to pass inspection the diameter x must lie in the interval in which $x_1 \leq x \leq x_2$. If $x < x_1$, the piece must be scrapped. If $x > x_2$, the piece can be reworked. The shop under consideration does not rework pieces. Instead, it sells pieces with $x > x_2$ to another shop at a price p, each, for rework. Each piece which passes inspection is sold at price $p > p_1$. The cost of raw materials, labor, and machine time for each piece which enters production is k. It is desired to determine the value of μ which maximizes the expected weekly profit.

If w pieces are machined per week, the expected number which must be scrapped is

$$w\int_0^{x_1} f(x;\mu)\,dx$$

[1]This example is taken from (19, pp. 58-60).

when the integral is simply the probability that the diameter of any piece will be less than x_1. Similarly, the expected number which will be sold for rework is

$$w\int_{x_2}^{x_m} f(x;\mu)\ dx$$

where x_m is the maximum diameter which any piece can have. Thus the expected weekly profit P is

$$P(\mu)=Pw[1-\int_0^{x_1} f(x;\mu)\,dx - \int_{x_2}^{x_m} f(x;\mu)\,dx]$$

$$+ P_1 w\int_{x_2}^{x_m} f(x;\mu)\,dx-wk$$

It is clear that the absolute maximum of $P(\mu)$ will not occur at the boundries where $\mu=0$ or x_m because these are not meaningful solutions. Thus, the value of μ when P takes on its maximum must be a solution to

$$\frac{dP(\mu)}{d\mu} =0-w[P\int_0^{x_1} \frac{\partial}{\partial\mu} f(x;\mu)\,dx+(P-P_1)\int_{x_2}^{x_m} \frac{\partial}{\partial\mu} f(x;\mu)\,dx]$$

In the case where x is normally distributed with mean μ and variance σ^2, one has the unique solution

$$\mu = \frac{x_1+x_2}{2} + \frac{\sigma^2}{x_2-x_1} k \ (\frac{P}{P-P_1})$$

Since the solution is unique, this value of μ must yield the absolute maximum of the expected weekly profit.

A method for obtaining the relative maximum or minimum values of a function $F(x,y,z)$ subject to a constraint condition $\phi(x,y,z)=0$, consists of the formation of the auxiliary function,

$$G(x,y,z) \equiv F(x,y,z) + \lambda\phi(x,y,z)$$

subject to the conditions

$$\frac{\partial G}{\partial x} = 0; \ \frac{\partial G}{\partial y} = 0; \ \frac{\partial G}{\partial z} = 0$$

which are necessary conditions for a relative maximum or minimum. The parameter λ, which is independent of x,y,z, is called a Lagrange multiplier.

The method can be generalized. If one wishes to find the relative maximum or minimum values of a function $F(x_1,x_2,\ldots,x_n)$ subject to the constraint conditions $\phi_1(x_1,\ldots x_n)=0$, $\phi_2(x_1,\ldots x_n)=0,\ldots\phi_k(x_1,\ldots x_n)=0$, we form the auxiliary function

$$G(x_1,x_2,\ldots x_n) \equiv F+\lambda_1\phi_1+\lambda_2\phi_2+\ldots\lambda_k\phi_k$$

subject to the (necessary) conditions

$$\frac{\partial G}{\partial x_1}=0; \quad \frac{\partial G}{\partial x_2}=0; \quad \ldots\ldots\frac{\partial G}{\partial x_n} \equiv 0$$

where $\lambda_1,\lambda_2,\ldots,\lambda_k$, which are independent of $x_1,x_2,\ldots,x_n$, are the Lagrange multipliers.

Here attention will be placed on solutions of inequality systems. When a decision problem requires the minimization of a linear form subject to linear inequality constraints, it is called a linear program. By natural extension, its study provides further insight into the problem of minimizing a convex function whose variables must satisfy a system of convex inequality constraints.

1.2. Models with Inequalities

Mathematical programming (48) is concerned with the problem of maximizing or minimizing a function of variables that are restricted by a number of constraints. Interest in this problem arose in economics and management sciences, where it was realized that many problems of optimum allocation of scarce resources could be formulated mathematically as programming problems. The introduction of large high-speed electronic computers, moreover, made it possible in principle to obtain numerical solutions, provided efficient mathematical methods and computational techniques could be developed. These methods cannot immediately be derived from classical tools, such as the method of Lagrange multipliers. The latter has effectively been applied to extremun problems in which the variables were only restricted by equality constraints but it is hardly, if at all, possible to extend such a method to inequality-constrained extremun problems. However, mathematical programming problems nearly always consist of many variables and constraints.

Mathematical programming has three aspects:

1. The application or technological problem, i.e. the form-

ulation of the model, the gathering of data, the interpretation and study of the results, etc.

2. The mathematical problem, i.e. the development of mathematical techniques for a certain class of models.
3. The computational problem, i.e. the study of the computational aspects of a mathematical method and the development of computer codes for it.

Mathematical programming problems can be divided into four classes:

1. Deterministic, continuous models; the set of points, satisfying all constraints-to be called feasible region-is connected; the objective function, i.e. the function to be optimized is continuous. In this class can be found:
   a. Linear programming, the following reference may be consulted (11).
   b. Quadratic programming, i.e. the problem of minimizing a convex quadratic function, subject to linear constraints.
   c. The problem of minimizing a general convex function, subject to linear constraints. Most of the method which was developed for this problem can be considered as large-step gradient methods.
   d. Convex programming, i.e. the problem of minimizing a convex function (or maximizing a concave function) in a convex region.

2. Deterministic, discontinuous models; the feasible region is not connected or (and) the objective function is not continuous. In this class one finds:
   a. Integer linear programming. The solution has to satisfy the additional requirement that it consists of integers.
   b. Mixed discrete continuous programming. Only part of the variables in the optimum solution must be integer-value. Many well-known case studies can be formulated as a mixed programming problem, e.g. the travelling-salesman problem and the fixed-charge problem.
3. Stochastic models; the coefficients in the constraints or (and) in the objective function are random variables. In this class one has the chance-constrained programming problems. A simple example is a linear programming problem with a stochastic requirements or objective vector.
4. Dynamic models; the coefficients in the constraints or (and) in the objective function are dependent on a parameter (e.g. the time). For each value of this parameter, it is desired to solve the problem. Dynamic models can often be solved by using Bellman's dynamic programming (7). In many cases the problem can also be formulated in a static way which may then give rise to a large programming problem.

Broadly speaking, mathematical programming problems deal with determining optimal allocation of limited resources to meet given objectives; more specifically, they deal with situations where a number of resources, such as men, materials, machines, and land, are available, and are to be combined to yield one or more products. There are, however, certain restrictions on all or some of the following broad categories, i.e.: on the total amount of each resource available, on the quantity of each product made, or on the quality of each product. Even within these restrictions there will exist many feasible allocations. Out of all permissible allocations of resources, it is desired to find the one or ones which maximize or minimize some numerical quantity, such as profit or cost.

2. PROGRAMMING MODELS UNDER RISK

In ordinary and usual linear programming (L.P.) problems

Max $c'x$

subject to

$Ax \leq b$

$x \geq 0$ (2.0.1)

It is assumed that all the parameters (i.e. the coefficients of the objective function), the inequalities, and the resource availabilities are exactly known without error. This assumption is relaxed when some or all elements of the set (c,A,b) are probabilistic, namely, the distribution approach of stochastic linear programming. Two special approaches, the decision-rule approach of chance-constrained programming, and the two-stage approach of programming under uncertainty, are available.

Most linear programming problems involve errors in either the input-output matrix, resource availabilities, or prices. Some of the more usual methods for reducing the effect of errors are:

1. Replacing the random elements by their expected values.
2. Replacing the random elements by pessimistic estimates of their values.
3. Recasting the problem into a two-stage problem in whose second stage one can compensate for inaccuracies in the first stage activities.

These methods are called the expected value solution, the "fat" solution, and the slack solution. (24)

The so-called "fat" formulation is characterized by the following reasoning. The decision-maker has to decide on some vector x of activities before he can observe the values of A and b. After he has made his choice, he is confronted with a particular A and b and can see whether or not x has satisfied the constraints. The difficulty, though, is that his prechosen x may not be feasible for the observed A and b. What the "fat" formulation prescribes is, that one restrict oneself to the convex set of those x which are feasible no matter what values of A and b will subsequently be observed.

A more realistic statement of the problem is what could be called the "slack" formulation. It involves converting the problem to a two-stage problem which can be described roughly as follows. The decision-maker is supposed to choose a non-negative x, then observe a value of the random matrix A and the random vector b, and finally compare Ax with b. The vector x may or may not be feasible. But whether feasible or not, one is going to allow the decision-maker after the fact, to make another decision y to compensate for discrepancies between Ax and b, based on his original decision x and the later-observed A and b, but at a penalty cost.

The linear inventory problem is an example of this kind. Here x is the amount of inventory which the storekeeper must have on hand, b is the later-to-be-observed random demand, A is a nonrandom matrix of relevant technology coefficients and y is the second-stage decision, embodying two kinds of activities. If the demand exceeds the inventory, the storekeeper must go out on the open market and at a penalty cost, buy goods to take care of the excess of demand over supply. If the inventory exceeds the demand, he will have to scrap the excess. This loss is a penalty due to not having made a better choice of x. This is a more realistic way of looking at the problem than the "fat" solution because it keeps the decision-maker in business after he has made his choice of x and the random variables have been observed. A simple example is considered in Dantzig (11).

2.1. Variation in Price Coefficients

The importance of correct specification of errors can be illustrated with respect to an ordinary linear program. Suppose one has variation in price coefficients. Then the problem will become

$$\text{Max } (c+\gamma)'x$$

Subject

$$Ax \le b \quad x \ge 0$$

Analytically, it is important to be able to give an economic interpretation to this type of error which may occur. Consider the error γ associated with the price coefficient which may originate in the following ways:

1. Realized prices may not correspond with planning prices.
2. The cost of production may change after the plan has been in operation.
3. An individual price depends essentially upon the yield of the activity.

In this case, if one supposes that there are no errors in the resource supplies and input-output coefficient, and if the errors γ are normally distributed with mean zero, then to maximize the expected value of objective function, we set γ equal to their expected values, i.e. $E(\gamma)=0$ and apply the standard simplex procedure to the original linear programming problem.

Freund (14) developed a model in which risk is taken into account in the selection of the optimum plan. Freund's model corresponds to the ordinary linear programming problem with the added generalization that it takes account of the variability of activities' net revenue due to sample variation in yield and price. He assumes that the risk aversion function takes the general form;

$$y = 1-e^{-\phi z}$$

where y is utility

z is the net revenue

ϕ is the risk aversion constant.

The larger the ϕ is, the greater the risk aversion. Freund then shows that this problem can be treated as a quadratic programming problem, and, as such, a solution can be obtained.

2.2. Stochastic Linear Programming

Stochastic linear programming attempts to deal with the situation in which the elements of one or more of the three sets of coefficients have a probability distribution as opposed to just being constants. The problem can be reformulated in the following manner. It is desirable to optimize (maximize or minimize)

$$F = (c+\gamma)'c$$

subject to the restrictions

$$(A+\alpha)x \leq (b+\beta)$$

where a_{ij}, c_j, b_i are some constants and $\alpha_{ijk}, \beta_{ik}, \gamma_{jk}$ are random variables with probability distribution in which

$$E\gamma_{jk}=\mu_{\gamma j}; \quad E\alpha_{ijk}=\mu_{\alpha ij} \quad E\beta_{ik}=\mu_{\beta i}$$

$$E(\gamma_{jk}-\mu_{\gamma j})^2=\sigma^2_{\gamma j}; \; E(\alpha_{ijk}-\mu_{\alpha ij})^2=\sigma^2_{\alpha ij}; \text{ and } E(\beta_{ik}-\mu_{\beta i})^2=\sigma^2_{\beta i}$$

These means and variances may not necessarily be known.

Let it be assumed (46) that it is known that the technological coefficients lie within given upper and lower limits and that

$$a^-_{ij} \leq a_{ij} \leq a^+_{ij}$$
$$b^-_j \leq b_j \leq b^+_j$$
$$c^-_i \leq c_i \leq c^+_i$$

where those values written with a minus or plus sign as superscript are known. It is then natural to ask what can be inferred as to the range of possible variation of the optimum of the objective function.

Vajda (46) has shown that

$$\underset{\Sigma a_{ij}^{+}x_i \geq b_j^{-}}{\text{Min } \Sigma c_i^{-} x_i} \;\leq\; \underset{\Sigma_i a_{ij}x_i \geq b_j}{\text{Min } \Sigma_i c_i x_i} \;\leq\; \underset{\Sigma a_{ij}x_i \geq b_j^{+}}{\text{Min } \Sigma c_i^{+} x_i}$$

and

$$\underset{\Sigma_i a_{ij}^{+}x_i \leq b_j^{-}}{\text{Max } \Sigma_i c_i^{-} x_i} \;\leq\; \underset{\Sigma_i a_{ij}x_i \leq b_j}{\text{Max } \Sigma_i c_i x_i} \;\leq\; \underset{\Sigma a_{ij}^{-}x_i \leq b_j^{+}}{\text{Max } \Sigma c_i^{+} x_i}$$

Stochastic linear programming consists of solving the ordinary linear program when it is given that the components of A,b, and/or c are no longer constants but rather variables with known and/or unknown probability distributions.

There have been four basic types of approach to this problem:

1. The probabilistic approach.
    a. Passive approach
    b. Active approach
2. Parametric approach.
3. Probabilistic-parametric approach.
4. Diversification approach.

The probabilistic approach is an empirical approach pioneered by Babbar (5) and Tintner (44). They have tried to estimate the probability distribution of the objective function, its optimal expectation, and the confidence interval about the expectation. Tintner (45) subdivides his approach into what he calls the "active" and the "passive" approaches.

The passive approach (also termed the "wait and see" approach) derives, by numerical methods if necessary, the distribution of (max $z=c'x$)(and other z's corresponding to basic solutions other than the optimal basic solution) under the assumption of a known probability distribution function of all the random parameters, i.e. (A,b,c) of the problem. This approach assumes that all admissible situations, i.e. for all admissible variations of the random parameters, the conditions of the simple nonstochastic linear program are fulfilled and the maximum achieved. The active or "here and now" approach to stochastic linear programming may be specified as follows:

Maximize $z = c'x$

under the conditions:

$Ax \leq BU$

where U is a matrix with m rows and n columns with elements μ_{ij}, such that

(m) $\mu_{ij} \geq 0 \quad \sum_{j=1}^{n} \mu_{ij} = 1$

when x is a diagonal matrix with elements of the vector x in the diagonal, and B is a diagonal matrix with the elements of the vector b in the diagonal. The probability distribution of (max z) will depend upon the allocation matrix $U=[\mu_{ij}]$ which defines a set of controlled (i.e., nonrandom) variables which may be appropriately chosen to optimize a risk preferred function (i.e., a utility function associated with the objective function). Let z_a denote the value of the objective function under the active approach and let U and $\tilde{U}$ represent two different sets of resource allocations that could be selected by the policy-maker (or the entrepreneur, for example, in a production situation). Since in every case all resources are to be fully allocated by condition (m), the selections of U and $\tilde{U}$ represent only different relative allocations for every resource $i=1,\ldots,m$. The resulting probability distribution for "max z_a" induced by these two allocations may then be compared for purpose of deciding upon the optimal allocation.

Sengupta (40) analyzes a method of characterizing the distribution of the objective function values corresponding to the set of extreme points in the solution space for both the active and the passive approaches. Truncation refers to the selection of extreme points that are neighbors, that is to say, to the optimal extreme point. The sensitivity of objective function values corresponding to truncated solutions is analyzed here in terms of stability properties, stability being measured in terms of variance. From an economic point of view, the approach outlined here offers a theory of the second best, since it specifies the set of conditions under which a value of the objective function, that corresponds to the optimum solution, on the average may have higher instability than another value of the objective function, that corresponds to a truncated solution, under the assumed conditions of stochastic linear programming.

The parametric approach (17) is a technique for dealing with stochastic variation in the coefficients of the objective function. Two sets of coefficients $[c'_j]$ and $[c''_j]$ are considered. A parameter t which can take on any finite values is introduced. The coefficients

$$c_j = c'_j + tc''_j$$

are used and the problem is dealt with by the usual simplex method. The values of the variables depend on the set of basic variables but not on the value of t which appears only in the objective function. If the solution space is considered, the choice of t means geometrically the choice of a preferred direction. Because there are constraints, there must be bounds on the feasible region. By varying t, it can be discovered where these bounds lie. This and other types of parametric approach are given in (29).

The probabilistic-parametric approach (Madansky (22)) considers a type of problem in which the constraints are not always met. Among all x and y whose probability of feasibility is at least P, it is desired to find the y which minimizes c'x+f'y and also to determine the value of x which minimizes E min (a'x+f'y) where E is the expectation operator.

Here f'y is the penalty paid for the deviation of the actual from the expected value c.

Madansky (24) also considers the problem where one wishes to maximize

$$\text{Prob}\ [\min_{y}\ (c'x+f'y) \leq k]$$

for some fixed preassigned k. Considering the case where only c is random, the suggestion was to replace the vector c by the vector c_γ where Prob $[c \leq c_\gamma] = \gamma$ and to solve the determinantal problem for x_γ. Then one could look for the largest γ and concomitant x_γ such that x_γ and $y(x_\gamma)$ are feasible with probability P or more and such that $F(c_\gamma x_\gamma) = k$. Unfortunately, in multi-dimensions c_γ is not unique and although x_γ is a continuous function of c_γ, it is not necessarily the case that by increasing γ Prob $[F(c_\gamma x_\gamma) \leq k]$ will increase.

In the diversification approach, Markowitz (25) dealt with the stochastic problem in a completely original manner. He proposed minimizing the variance of these coefficients for their given expected values or alternatively maximizing their expected values for a given variance.

In a standard stochastic problem, the coefficients are usually mean values of sample means and are not greatly different from the population means. It is Tchebycheff's inequality which states that

$$\text{Prob}\ (|\bar{x}-\mu|>b)\ <\ \frac{\sigma^2}{nb^2}$$

This fact must be taken into consideration when the confidence region of the objective function is calculated.

Babbar (4) has gone into some theoretical detail in deriving the general case for the distribution of the objective function when all three sets of coefficients are stochastic. But he concluded that unless the elements have normal distributions, the problem of obtaining the distribution of the objective function and a confidence region about its expected value become unmanageable in most cases.

Application to economic models of stochastic linear programming will be found in Morrison (29).

2.3. Chance-Constrained Programming[1]

A new conceptual and analytical vehicle for problems of temporal planning under uncertainty, involving determination of optimal (sequential) stochastic decision rules is defined by Charnes and Cooper (8).

The problem of stochastic (or better, chance-constrained) programming is defined as follows. Select certain random variables with known distributions in such a manner as (a) to maximize a functional of both classes of random variables subject to (b) constraints on these variables which must be main-

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[1]This part is based on the papers by Charnes and Cooper (9), J. K. Sengupta (13, Chapter 9), Kataoka (21), J. K. Sengupta (37).

tained at prescribed levels of probability. More loosely, the problem is to determine optimal stochastic decision rules under these circumstances. An example is supplied in (10). Temporal planning in which uncertainty elements are present, but in which management has access to "control variables" with which to influence outcomes, is a general way of characterizing these problems. Thus, queuing problems in which the availability of servers, customers, or both are partly controllable fall within this classification. It should be noted that the constraints to be maintained at the specified levels of probability will typically be given in the form of inequalities.

Chance-constrained programming admits random data variations and permits constraint violations up to specified probability limits. Different kinds of decision rules and optimizing objectives may be used so that under certain conditions, a programming problem (not necessarily linear) can be achieved, that is deterministic in that all random elements have been eliminated. Existance of such "deterministic equivalent" in the form of specified convex programming problems is established for a general class of linear decision rules (9) under the following three classes of objectives: (1) maximum expected value ('E model'); (2) minimum variance ('V model') and (3) maximum probability ('P model').

A chance-constrained formulation would replace the ordinary linear programming problem with a problem of the following kind:

Optimize $f(c,x)=\text{Max } c'x$

Subject to $\text{Prob } (Ax\leq b)\geq\alpha, \quad x\geq 0$ (2.3.1)

A,b,c are not necessarily constant but have, in general, some or all of their elements as random variables. The vector α contains a prescribed set of constants that are probability measures of the extent to which constraint violations are admitted. Thus, an element $0\leq\alpha_i\leq 1$ is associated with a constraint $\sum_{j=1}^{n} a_{ij}x_j\leq b_i$ to give

$$\text{Prob } \left(\sum_{j=1}^{n} a_{ij}x_j\leq b_i\right)\geq\alpha_i \quad (2.3.2)$$

a double inequality which is interpreted to mean that the i^{th} constraint may be violated but at most $\beta_i=1-\alpha_i$ proportion of the time.

Here it is proposed to examine important classes of chance-constrained problems and to obtain deterministic equivalents that are then known in certain cases to be convex programming problems. It is to be emphasized, however, that optimization under risk immediately raises very important questions concerning a choice of rational objectives. Questions can arise, for example, concerning the reasonableness of an expected value optimization. Without attempting to resolve these issues, it should be noted that the evaluators secured for one objective are not necessarily correct or optimal when applied to the same problem under an altered objective.

It is assumed that a choice of values for decision variables x will not disturb the densities associated with the random variables in A,b,c. Then we may formulate the general problem in terms of choosing a suitable decision rule

$$x = \phi(A,b,c) \qquad (2.3.3)$$

with the function ϕ, to be chosen from a prescribed class of functions and applied in a manner that guarantees that x values, as generated, will satisfy the chance constraints of (2.3.1) and optimize f(c,x) in (2.3.1) with reference to the class of rules from which the ϕ of (2.3.2) is to be chosen.

By assuming that the matrix A is constant (i.e. non-random) I will also be restricted by the rule (2.3.3) to members of the class

$$x = Db \qquad (2.3.4)$$

where D is an n x n matrix whose elements are to be determined by reference to (2.3.1).

We will examine all possible rules of form D and, for important classes of objective and statistical distributions, in order to be able to characterize situations in which a deterministic equivalent will be achieved-irrespective of the D choice thus yielding a convex programming problem.

The expected value model ('E model') is then

maximize $\quad E\, c'\, x$

under conditions $\quad \text{Prob}\,(Ax \leq b) \geq \alpha \qquad (2.3.5)$

$$x=Db$$

substituting (2.3.4) into the objective function of (2.3.5) one obtains

$$E(c'Db) = (Ec)'\ D(Eb)$$

it will assume that b and c are uncorrelated, then it will define the vectors

$$\mu_c' \equiv (Ec)' \ ; \quad \mu_b' \equiv (Eb)'$$

then

$$\text{Min} - \mu_c' D\mu_b$$

Denoting the i^{th} row of the matrix A by a_i' and $(b-\mu_b)$ by $\hat{b}$ and assuming normality of distribution for the variates $(a_i', D\ \hat{b} - \hat{b}_i)$, parts of the constraints of (2.3.4) may be written as

$$\text{Prob}\ (a_i'Db - b_i \leq 0) = \text{Prob}\ (b_i - a_i'Db \geq 0)$$

$$= \text{Prob}\ (\hat{b}_i - a_i'D\hat{b} \geq -\mu_{b_i} + a_i'D\mu_b) \geq \alpha_i$$

Assuming $E(\hat{b}_i - a_i'D\hat{b})^2 > 0$, the above can be normalized and i^{th} constraint can be written fully as

$$\text{Prob}\left[\frac{\hat{b}_i - a_i'\ D\ \hat{b}_i}{\sqrt{E(\hat{b}_i - a_i\ D\ \hat{b})^2}} \geq \frac{-\mu b_i + a_i'\ D\ \mu_b}{\sqrt{E(\hat{b}_i - a_i\ D\ \hat{b})^2}}\right] \geq \alpha_i \qquad (2.3.6)$$

by the assumption of normality, the left-hand side of the argument, i.e. $(\hat{b} - a_i'D\ \hat{b}_i) \ / \ \sqrt{E\ (\hat{b}_i - a_i'D\ \hat{b})^2}$ is a standardized normal variable with zero mean and unit variance, so that (2.3.6) is replaced by

$$F_i \left| \frac{-\mu_{b_i}+a_i{}'D\ \mu_b}{\sqrt{E(\hat{b}_i - a_i{}'\ D\ \hat{b})^2}} \right| \geq \alpha_i \qquad (2.3.7)$$

where

$$F_i(w) = (\sqrt{2\pi})^{-1}\int_w^\infty e^{-y^2/2}\ dy$$

Usually for normal distribution $\alpha_i \geq 0.5$ is taken, then the equation (2.3.7) can be solved as

$$\frac{-\mu_b+a_i{}'\ D\ \mu_b}{\sqrt{E(\hat{b}_i-a_i{}'\ D\ \hat{b})^2}} \leq F_i{}^{-1}(\alpha_i) \equiv -q_i \qquad (2.3.8)$$

where $q_i>0$ for all i, if $\alpha_i>0.5$.

The system (2.3.8) which involves nonrandom variables (i.e.), deterministic values only can be further reduced to a convex programming problem by introducing new variable v_i and writing (2.3.8) as

$$-\mu_{b_i}+a_i{}'D\mu_b \leq -v_i \leq -q_i \quad \sqrt{E(\hat{b}_i-a_i{}'D\ \hat{b})^2} \leq 0$$

or

$$\mu_{b_i}-a_i{}'D\ \mu_b \geq v_i \geq q_i \quad \sqrt{E(\hat{b}_i-a_i{}'D\ \hat{b})^2} \geq 0$$

which can be further simplified by squaring both sides, since nonnegativity is assigned to all expressions between inequality signs i.e.,

$$-a_i{}'D\mu_b-v_i \geq -\mu_{b_i}$$

$$-q_i{}^2\ E(\hat{b}_i-a_i{}'Db)^2+v_i{}^2 \geq 0$$

with $v_i \geq 0$ for each i. Hence, the equivalent convex program for chance-constrained programming (2.3.5) is

Minimize $-\mu_c{}' D \mu_b$

under the conditions

$$\mu_{b_i} - a_i{}' D \mu_b - v_i \geq 0 \qquad (2.3.9)$$

$$-q_i{}^2 E(a_i{}'Db-b_i)^2 + q_i{}^2(\mu_{b_i} - a_i{}'D\mu_b)^2 + v_i{}^2 \geq 0$$

where the problem (2.3.9) is a convex programming problem in the variables D and v

For the minimum variance ('V model')

Min $E(c'x-c^{0}{}'x^{0})^2$

under the conditions (2.3.10)

Prob $(Ax \leq b) \geq \alpha$

$x=Db$

where the objective is to minimize a generalized mean square error i.e. taking all relations between the c_j into account, it is intended to minimize this measure of their deviations about some given preferred values $z^{0}=c^{0}{}'x^{0}$.

It is easy to achieve the following deterministic equivalent to (2.3.10)

Min $E(c'Db-c^{0}{}'x')^2$

under conditions

$$\mu_{b_i} - a_i{}'D\mu_b - v_i \geq 0$$

$$-q_i{}^2 E(a_i{}'Db-b_i)^2 + q_i{}^2(\mu_{b_i} - a_i{}'D\mu_b)^2 + v_i{}^2 \geq 0$$

$$v_i \geq 0$$

This deterministic equivalent is again a convex programming problem.

The maximum probability ('P model') turns to a version of the satisficing approach. In this approach the $c^{0\prime}x^0$ components are specified relative to some set of values - e.g. as generated from an aspiration level mechanism - which an organization (an individual or a business firm in the present context) will regard as satisfactory whenever these levels are achieved. Of course, when confronting an environment subject to risk, the organization cannot be sure of achieving these levels when effecting its choice from what it believes are available alternatives. On the other hand, if it does not achieve the indicated $c^{0\prime}x^0$ levels or, more precisely, if it believes that it cannot achieve them at a satisfactory level of probability, then the organization will either (a) reorient its activities and 'search' for a more favorable environment or else (b) alter its aspirations and, possibly, the probability of achieving them.

The model is Max Prob $(c'x \geq c^{0\prime}x^0)$

under the conditions

$$\text{Prob}\,(Ax \leq b) \geq \alpha \qquad (2.3.11)$$

$$x = Db$$

If the same rules and assumptions are utilized as before to reduce this to a deterministic equivalent, it then becomes

$$\text{Max } v_0/w_0$$

under conditions

$$\mu_c{}'D\ \mu_b - v_0 \geq \mu_{c^0} \tag{2.3.12}$$

$$-E(c'Db-c^{0}{}'x^0)^2+W_0{}^2 \geq 0$$

$$\mu_{b_i}-a_i{}'D\ \mu_b-v_i \geq 0$$

$$-q_i{}^2\ E(a_i{}'Db-b_i)^2+q_i{}^2(\mu_{b_i}-a_i{}'D\ \mu_b)^2-v_i{}^2 \geq 0$$

$$v_i \geq 0$$

This problem can be solved using fractional programming methods; for more details see (9).

Sengupta (13) points out two aspects which may be noted about this method. The first aspect is that it characterizes the problem only within a very restricted class of decision rules, and the operational efficiency of the method must be determined by further experimentation. In other words, one could specify other types of deterministic equivalents (6) which would subsume the cases considered here. Secondly, the decision rules here are not analytic, i.e., each time they have to be solved with the appearance of new data. An extension of this idea of deterministic equivalent in terms of recursive programming may be helpful, although it will involve nonlinear difference equations that are very difficult to solve.

Shinji Kataoka (21) introduced a new objective function, which is suitable for stochastic programming, utilizing Charnes' and Cooper's model. That is

$$\text{Max } f \qquad (2.3.13)$$

Subject to $\quad \text{Prob } (c'x \leq f) = \alpha \qquad (2.3.14)$

and $\quad \text{Prob } (Ax \leq b) \geq \beta \qquad (2.3.15)$

$$x \geq 0$$

It should be noted that the expected value of profit is not always considered a good measure for the optimality criterion. Even though a policy x dominates other policies in the expectation of profit, it may be more risky in that the chance of getting a very low profit may be greater than for other policies because of the dispersion of its distribution. For this reason, the lower allowable limit f defined by (2.3.14) a special form of (2.3.15) for a given probability α is maximized instead of the expected value profit.

A case is considered in which the b_i's and c_j's are random variables, but the a_{ij}'s are constant. Transportation and production horizon problems belong to this category if customer demand and commodity price are random. This is called a transportation type problem.

Kataoka has made the following assumptions and formulations.

A.1. The random variable b_i has a normal distribution with mean value $\bar{b}_i$ and variance $\sigma_{b_i}^2$

The probability in (2.3.15) can be transformed as

$$\text{Prob}\ (\sum_j a_{ij}x_j \leq b_i) = \text{Prob}\ (\frac{b-\bar{b}_i}{\sigma_{b_i}} \geq \frac{\sum_j a_{ij}x_j - \bar{b}_i}{\sigma_{b_i}})$$

then the left hand side of the argument is a normalized random variable with zero mean and unit variance. Hence the probability condition, (2.3.15) is replaced by

$$G(\frac{\sum_j a_{ij}x_j - \bar{b}_i}{\sigma_{b_i}}) \geq \beta_i$$

or

$$\sum_j a_{ij}x_j - \bar{b}_i \leq G^{-1}(\beta_i) = -q_i$$

where

$$G(x) = (\sqrt{2\Pi})^{-1} \int_x^{\infty} e^{-y^2/2}\, dy$$

usually it is considered that $\beta_i \geq 0.5$; then $G^{-1}(\beta_i) \leq 0$.

A.2. The vector c has a multinormal distribution with mean value vector $\bar{c} = (c_1, c_2, \ldots, c_n)$ and a dispersion matrix V. The variance of $c'x$ is $x'Vx$. Hence

$$\text{Prob}(c'x \leq f) = \text{Prob}\ (\frac{c'x - \bar{c}x}{\sqrt{x'vx}} \leq \frac{f - \bar{c}'x}{\sqrt{x'vx}}) = I(\frac{f - \bar{c}'x}{\sqrt{x'vx}})$$

where

$$I(x) = (\sqrt{2\Pi})^{-1} \int_{-\infty}^{x} e^{-y^2/2}\, dy$$

then for (2.3.14) is

$$f = \bar{c}'x + I^{-1}(\alpha)\sqrt{x'vx}$$

Finally Kataoka has a maximization problem

$$\text{Max } f_I=\bar{c}'x+I^{-1}(\alpha)\sqrt{x'Vx} \qquad (2.3.16)$$

under the conditions

$$Ax\leq\bar{b}_i+G^{-1}(\beta_i)\sigma_{b_i}$$

Kataoka also transforms a model to a more general stochastic programming problem in which the components of matrix A are random variables; for more details see (21).

Sengupta (37) considers three generalized standpoints. First, the assumption of normality is replaced by a chi-square distribution, which has a nonnegative range and hence more applicability to economic problems of production planning; and a confidence interval for the optimal solution vector is worked out on this basis. Second, the relevance of chance-constrained programming to sensitivity analysis of optimizing economic models is briefly indicated. Third, the applicability of chance-constrained decision rules to problems of development planning through investment programming is discussed.

Sengupta (37) assumes that the elements a_{ij}, b_i, of A and b respectively are taken to be mutually independent chi-square variates with means $\bar{a}_{ij}$ and $\bar{b}_i$ and these are denoted by χ^2_{ij} (a_{ij}) and $\chi^2(b_i)$ respectively. He mentions two points about the reasonableness of this assumption. First, in most economic problems of production and resource allocation, the input coefficients a_{ij} represent coefficients of production function and therefore these must be nonnegative.

Similarly, the resource vector must be nonnegative. Second a chi-square, which is closely related to the normal (e.g., a normal variate truncated at $y \geq 0$ results in a chi-square) has properties very similar to a normal distribution (e.g., reproductive properties) and hence approximations can easily be worked out by means of normal tables whenever needed.

In the derivation of his model, Sengupta assumes for a moment that b is not random. By transformation (2.3.1) becomes

$$\text{Prob}\left(\chi_i^2\left(\sum_{j=1}^{n} \bar{a}_{ij}\right) \leq \frac{b_i \sum_j \bar{a}_{ij} x_j}{\sum_j \bar{a}_{ij}\, x_j^2}\right) \geq \alpha_i \qquad (2.3.17)$$

or, alternatively as,

$$F_i\left(b_i \sum_{j=1}^{n} \bar{a}_{ij} x_j \Big/ \sum_{j=1}^{n} \bar{a}_{ij}\, x_j^2\right) \geq \alpha_i \qquad (i=1,\ldots n)$$

where f_i (w) is the cumulative distribution function of a central chi-square variate with degrees of freedom $N=\sum_j \bar{a}_{ij}$, i.e.

$$F_i(w)=\left(2^{n/2}\, \Gamma(N/2)\right)^{-1} \int_0^w t^{(n/2)-1}\cdot \exp(-t/2)\, dt$$

Since the ordinary chi-square tables give the various significance points for w for a given degree of freedom, it would be possible to compare the exact values of

$$w= b_i \sum_{j=1}^{n} \bar{a}_{ij} x_j \Big/ \sum_{j=1}^{n} \bar{a}_{ij} x_j^2$$

satisfying the inequality (2.3.17). For example, if d_i=.990 (i.e. the tolerance measure) and $\sum_j \bar{a}_{ij}$=7.0, then from the chi-square table one finds that

$$\text{Prob}(\chi_i^2\ (7.0) \leq w_0) = .990$$

implies a value of w_0=18.4753. Therefore, if it is taken that $w \geq w_0$, this would satisfy a tolerance measure of 99% or higher. Since, for any preassigned value of tolerance measure α_i and the value of $N=\sum_j \bar{a}_{ij}$, one can find a positive value of w_0 from the chi-square table.

The chance-constrained programming model (2.3.1) then is finalized as a convex programming problem of the following type.

$$\text{Minimize } -c'x = -\sum_{j=1}^{n} c_j x_j$$

under the conditions

$$b_i \sum_{j=1}^{n} \bar{a}_{ij} x_j - q_i \sum_{j=1}^{n} \bar{a}_{ij} x_j^2 \geq 0$$

where

$$q_i = w_0 \qquad x_j \geq 0$$

For a general case, Sengupta uses the F distribution when b is also random and he obtains the following concave programming problem.

$$\text{Maximize } c'x = \sum_{j=1}^{n} c_j x_j$$

under the restrictions

$$\bar{b}_i \left(\sum_{j=1}^{n} \bar{a}_{ij} x_j\right) - k_i \left(\sum_{j=1}^{n} \bar{a}_{ij} x_j^2\right) \left(\sum_{j=1}^{n} \bar{a}_{ij}\right) \geq 0$$

$$x_j \geq 0$$

where K_i is obtained as follows.

$$r=\left(\sum_{j=1}^{n}\bar{a}_{ij}x_j^2\right)\left(\sum_j \bar{a}_{ij}\right)/\bar{b}_i\left(\sum_{j=1}^{n}a_{ij}x_j\right)$$

$$M_1=\sum_{j=1}^{n}\bar{a}_{ij};\ M_2=\bar{b}_i$$

therefore

$$\text{Prob}\ (F(M_1,M_2)\ \leq 1/r)\ \geq \alpha_i$$

then

$$K_i=1/r_0$$

Sengupta (37) considers that at the macroeconomic level, chance-constrained interpretations are most appropriate for the restrictions of a linear programming model applied to development planning. At the microeconomic level, the chance-constrained model is applicable most appropriately to situations of portfolio investment allocation and the holding of assets when a margin of safety is desired.

Further interesting results can be obtained assuming another kind of distribution with nonnegative range such as the exponential, the gamma or the beta distribution.

2.4. Safety First Principle

In the economic world disasters may occur. For a great many people, the idea of a disaster exists and the principle of "safety first" asserts that it is reasonable and probable in practice that an individual will seek to reduce as far as possible the chance of a catastrophe occurring.

A single disaster is a discontinuity in one's pattern of behavior and in one's scale of preferences, viz. death, bankruptcy or a prison sentence.

A. D. Roy (33) has developed the safety first principle in terms of minimizing the upper bound of the chance of a dread event, where the information available about the joint probability distribution of future occurrences is confined to the first and the second order moments only.

From a formal standpoint, the minimization of the chance of a disaster can be interpreted as maximizing expected utility if the utility function assumes only two values, e.g. one if disaster does not occur, and zero if it does. It would appear that this formal analogy is scarcely helpful, since in the one case an individual is trying to make the expected proportion of occurrences of disaster as small as possible, while in maximizing expected utility he is operating at a different level of satisfaction.

A complete hypothesis about individual or corporate economic behavior under uncertainty must specify three things. It must describe the way in which expectations are formed from experience of the hard facts of life, the objectives which the entity under examination is trying to achieve, and the opportunities present for attaining such ends.

It may be possible that the outcome of economic activity, which is regarded as disaster, is not independent of the expected value of the outcome. Thus, a person may be prepared to revise the level of disaster downwards if the expected return is at the same time raised. For example, he may at one and the same time regard a sepculative loss of 10 percent as a disaster if the expected gain is only 5 percent, while, if the expected gain is 15 per cent, he will only get excited if his loss exceeds 25 per cent. Once again, such individual psychology can no doubt be interpreted in terms of utility function, but such development will not be pursued here. In the following discussion, the disaster level of the outcome is taken to be constant.

Let it be supposed, then, that the principle of safety first is adopted and that, when confronted with a range of possible actions, we are concerned that our gross return m should not be less than some quantity d. With every possible action, this outcome is not certain. There is coupled with m a quantity σ (the standard error of m) which is, very roughly, the average amount by which the prediction m is expected to be wrong. In the following, it is assumed that m and σ are known precisely, whereas in fact they must be estimated from information about the past. This raises all kinds of problems, which are beyond the scope of this discussion, since estimates of m and σ, say $\hat{m}$ and $\hat{\sigma}$, will themselves have sampling distributions. Thus a full analysis of the problem should discuss simultaneously not only behavior under uncertainty but also actions under uncertain uncertainty.

In the particular application of the principle of safety first which is examined here, (33), it is postulated that m and σ are the quantities that can be distilled out of our knowledge of the past. The slightest acquaintance with problems of analyzing economic time series will suggest that this assumption is optimistic rather than unnecessarily restrictive.

Given the values of m and σ for all feasible choices of action, there will exist a functional relationship between these quantities, which will be denoted by $F(\sigma,m)=0$. There may be a whole family of such relationships; in this case $F(\sigma,m)=0$ is their envelope. Since it is not possible to determine with this information the precise probability of the final return being d or less for a given pair of values of m and σ, the only alternative open is a calculation of the upper bound of this probability. This can be done by an appeal to the Bienaymé-Tchebycheff inequality. Thus, if the final return is a random variable z then

$$\text{Prob}\ (|z-m| \geq m-d) \leq \frac{\sigma^2}{(m-d)^2}$$

If, then, in default of minimizing $P(z \leq d)$, one operates on $\sigma^2/(m-d)^2$, this is equivalent to maximizing $(m-d)/\sigma$.

Telser (43) postulates a particular attitude toward risk with stems from Roy's paper dealing with the theory of asset holding. He asks what assumptions make about about entrepreneurial behavior in the face of uncertainty and whether or not entrepreneurs maximize their expected income. Suppose an entrepreneur wishes to select a portfolio of assets so as to maximize expected net income. Then he would buy only one asset, namely, that whose price is expected to increase the most. If he is right, he would gain a great deal, but conversely, if he is wrong he would lose a great deal. It has been observed that people diversify their portfolios, hence reject the hypothesis that entrepreneurs maximize expected net income.

However, entrepreneurs do prefer larger net incomes to smaller net incomes. Suppose an entrepreneur considers all his actions and strategies and for each action calculates the probability that the income resulting from the action, which is a random variable, falls short of a disaster level. For each action a there is a probability distribution of net income I which can be written Prob $(I \leq c;\ a)$ where c is some constant. One computes the Prob $(I \leq r;\ a) = p$, where $o \leq p \leq 1$, and r is the disaster level of income. This disaster level of income, r, could be associated with bankruptcy or with something less dramatic.

Suppose that the entrepreneur does not want the probability of his net income falling short of r to exceed α. Hence he will not choose any action such that Prob $(I \leq r;\ a) = p > \alpha$. By this means, all his actions can be put into one of two classes. The first class consists of all the actions a such that Prob

$(I \leq r;\ a) > \alpha$ and the second class consists of all the actions a such that the Prob $(I \leq r;\ a) \leq \alpha$. All the actions in the second class shall be called admissible.

Then the entrepreneur will choose that action a of the admissible actions such that his expected income is at a maximum. Mathematically this means that the entrepreneur chooses the action a so that:

$$\underset{a}{\text{Max}}\ \Sigma\ I$$

Subject to

$$\text{Prob } (I \leq r;\ a) \leq \alpha$$

It would appear that such a rule of behavior requires that the entrepreneur knows the probability distribution of I for any action a that he might choose.

Fortunately we may appeal to the Tchebycheff inequality which permits one to set an upper bound to the Prob $(I \leq r;\ a)$ even when one does not know the probability distribution of I. The Tchebycheff inequality permits one to assert that:

$$\text{Prob } (|I-\bar{I}| \geq K) \leq \frac{\sigma^2}{K^2}$$

where $K>0$, σ^2=variance of I and $\bar{I}$=mean of I

It is not hard to show that

$$\text{Prob } (I \leq r) \leq \frac{\sigma^2}{(\bar{I}-r)^2}$$

This means that when $\frac{\sigma^2}{(\bar{I}-r)^2} \leq \alpha$ then Prob $(I \leq r) < \alpha$

Accordingly,

$$\frac{\sigma^2}{(\bar{I}-r)^2} \leq \alpha$$

becomes the risk restriction which is used.

It is assumed that the entrepreneur knows σ^2 and $\bar{I}$ for each a, but that he knows nothing more about the probability distribution of I for each a.

This formulation of the safety - first principle differs from that of A. D. Roy. He assumes that entrepreneurs minimize the probability of disaster. If they did, then their expected net income for that action which minimized the probability of disaster could be less than zero, i.e. they could be expected to lose money on their portfolio. This implies that there is no asset which the entrepreneur can hold without risk, that is, without the chance of gain or loss.

Sengupta (39) attempts to generalize the decision rules under chance-constrained programming from the viewpoint of safety first principles based on Tchebycheff-type proabbilistic inequalities. The latter inequalities are utilized to define distribution free tolerance levels. The optimization criterion of chance-constrained programming based on the mean and variance is extended to a more generalized formulation based on the Kolmogorov-Smirnov's statistic on the maximum discrepancy of the population and sampling distributions.

3. PARAMETRIC MODELS AND THE SENSITIVITY ANALYSIS APPROACH

We are treating two categories of problems relating to the same general question: What is the effect on the solution of a change in the given data of a problem? This question may arise after an optimal program has been found, but may equally arise at the beginning, if one wishes to explore the set of optimal programs by considering certain data as parametric variables. More specifically, we shall call problems of post-optimization those in which definite modification of given data is made in the matrix of coefficients A, the requirements vector b or the cost or profit c. We shall call parametric problems those in which the data vary in a continuous manner; then the problem is to study the variation of the optimal program as a function of the (variable) values of certain data.

In its most general form, in which the data varies as an implicit function of several independent parameters of arbitrary degree, this problem has not been solved. The only case which is really well known is that where the parameters occur in the first degree, especially where a single parameter occurs linearly in b or c.

In the formulation and solution of linear-programming problems, one essentially assumes at least initially, that all values of the coefficients are given and exact. Actually, such coefficients are derived from analysis of data and usually represent average values or best-estimate values. Accordingly, it is most important to analyze the sensitivity of the solution to variations in these coefficients or in the estimates of these coefficients. Stated still another way, one seeks to determine the range of variation of the coefficients over which the solution will remain optimal. Sensitivity studies of this sort are known as parametric linear programming.

Without a knowledge of the probability distributions of the coefficients, questions regarding sensitivity of solutions can presently be answered only in a limited sense. As noted by Gass (15, p. 123) not much has been accomplished to date with respect to sensitivity analysis for variations in the coefficients in the matrix of a_{ij} and detailed study of the effects of variations of either the objective function cost coefficients or the constant on the right-hand side has been limited to special cases. Needless to say, much research remains to be done in the area of parametric programming.

3.1. Parametric Programming and Sensitivity Analysis

Methods of sensitivity analysis which concentrate on the optimum set of basic activities (i.e., optimum solution vectors x^o and y^o) may be appropriately called parametric programming, since they essentially consider the set of restrictions to be placed on the variation of the parameters, (A,b,c,) such that the optimum activity-mix x^o say still retains its optimum character.

A major task in the development of realistic linear programming models is the gathering of accurate and reliable numerical values for the coefficients. Hence, it is important to study the behavior of solutions to linear programming problems when the coefficients of that problem are allowed to vary. This type of investigation is the function of parametric linear programming.

Once some linear programming problem of practical interest has been solved, we may discover that one or more of the prices were incorrect, one or more of the b_i were wrong, and perhaps a decimal point was misplaced in some a_{ij}. It may even turn out that some variable of interest or some constraint was omitted from this problem.

It is the purpose to show how to keep to a minimum the additional computational effort required to take care of above problems. In many cases, it is not necessary to solve the problems over again. A relatively small amount of work applied to the optimal solution will suffice. In other cases, however, there is no alternative but to go back to the beginning and resolve the problem.

There are seven specific problems. These can be briefly stated as follows.[1]

1. How much can the price vector c be changed in some specified way before the optimal solution obtained will no longer be optimal?
2. For a given change in c, how do we proceed to a new optimal solution if the original solution is no longer optimal?
3. How much can the requirements vector b be changed in some special way before the optimal solution will no longer be feasible?
4. If a given change in b makes the optimal solution no longer feasible, how do we proceed to a new optimal solution?
5. How can the addition of another variable (vector) be accounted for?
6. How can the insertion of an additional constraint be incorporated into the system?
7. Changes in the matrix elements a_{ij}.

The technique of how to handle these problems is given in (2,15,18).

Consider the problem of allocating labor to different jobs. The labor available is a variable function of time. In Saaty (34), a schedule of allocating labor (in the shipping

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[1]Most of this section has been taken from G. Hadley (18, Chapter 11).

operation) whose available amount is a function of time, to different tasks, in order to minimize the total cost, is given. The problem is cast in linear programming form in which all the coefficients are parameterized. The dependence of the optimal value on the parameterized coefficients leads to a sensitivity study.

Perhaps the most important operational approach of sensitivity analysis arises when we consider the sensitivity of the extreme value of the objective function in the neighborhood of the optimum by obtaining a series expansion for the objective function (38). Denote the primal and dual problems in standard matrix notation as

Primal: $\text{Max } F = c'x$

subject to

$$Ax \leq b \qquad x \geq o$$

Dual: $\text{Min } W = y'b$

subject to

$$A'y \geq c \qquad y \geq o$$

where x and y are column vectors of n components, A is a matrix of m rows and n columns and prime over a variable denotes transposition. Now assuming the above to be a regular linear programming problem (i.e., abstracting from degeneracy and other peculiarities), let x^o and y^o be the optimal solution vectors respectively with the associated set A^o, c^o, b^o.

Denoting by v the common value $F^o = W^o$ and then following Saaty's (35) procedure one could derive easily the following partial derivatives:

$$\frac{\partial v}{\partial c^o} = x^o \qquad \text{(i)}$$

$$\frac{\partial v}{\partial b^o} = y^o \qquad \text{(ii)}$$

$$\frac{\partial v}{\partial a_{ij}} = -x_j^o y_i^o \qquad \text{(iii)}$$

where $A = (a_{ij})$

provided, of course, such expansions around the optimal point (x^o, y^o) are valid, i.e., the vector c has to be in the interior of the cone associated with the solution vertex. These sensitivity indices have been further generalized by considering the optimal value $v = F^o = W^o$ as a function of a vector of parameters, say time t in its different phases. Averaging of such indices over a series of steady-state time periods gives a method of evaluating changes in the neighborhood of the optimal objective function. As Webb has remarked on the operational implications of these sensitivity indicators:

> These practical results are of value in determining the required accuracy of basic data systems, evaluating the significance of management changes in parameters, determining most significant parameters and the detecting of trend in the operation.

Sengupta (38) said that two things must, however, be pointed out about such a type of sensitivity indicator, especially the relation (iii) appropriately sealed. First it may offer a great help by way of developing working rules for screening a set of observed data on the coefficients a_{ij}, just like the statistical rules for rejection of outliers in practical work of statistical estimation. Secondly, these indicators dependent as they are on the duality theorem of linear-programming are not necessarily such that they can be applied to any basic feasible solution (or the objective function corresponding to it) other than the optimal basic feasible solution. In other words, this type of sensitivity analysis is strictly applicable to the optimal objective function and the associated optimal solution vectors x^o, y^o. Hence, when it is possible to wait and see the range of observed variation in the input-coefficients and then pick the optimal pair (x^o, y^o) for a specific a_{ij} or a collection of them, the above type of sensitivity analysis, partial as they are, may be of great help.

3.2. Range Analysis

Let us point out Le Chatelier's (36) principle which has the following statement:

> If the external condition of a thermodynamic system is altered, the equilibrium of the system will tend to move in such a direction as to oppose the change in external conditions.

An extension of Le Chatelier's principle is as follows: in linear programming problems, for any small change in the cost coefficients c_i the change in x_i will be smaller every time a new constraint is added to the system.

As noted by Gass (15) the investigation of parametric programming as applied to the variation of the coefficients of the objective function originated in the study of a dynamic (multiperiod) product inventory problem in which a manufacturer of a seasonal item must determine optimum monthly production schedules, so that customer demand can always be satisfied by a combination of current production and overproduction (i.e., inventory) from previous months. Here, one seeks to minimize the sum of costs due to output fluctuations (e.g., overtime, hiring and layoff, etc.) and to inventories.

One parameter linear programming program as considered by Gass and Saaty (17) may be stated mathematically as:

Let $\delta \leq \lambda \leq \phi$ where δ is any arbitrary, algebraically small, but finite number and ϕ is any arbitrary, algebraically large, but finite number. For each λ in this interval, find a vector $x = (x_1, x_2 \ldots x_n)$ such that

$$\text{Min} \sum_{j=1}^{n} (c_j + \lambda c'_j) x_j \qquad (3.3.1)$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij} x_j = b_i \qquad (i=1,\ldots m)$$

$$x_j \geq 0 \qquad (j=1,\ldots n) \quad (3.3.2)$$

where c'_j, c_j, a_{ij} and b_i are constants.

Let's assume that this problem is non-degenerate and that a basic feasible solution of equation (3.3.2) is already available. Then solving their problem by the simplex technique we have two cases:

1. A solution exists for $\lambda=\delta$. The optimality-criterion function z_j-c_j can be represented as a linear function of λ, namely

$$z_j-c_j = \alpha_j+\lambda\beta_j$$

Hence, for an optimum solution for $\lambda=\delta$, one must have

$$\alpha_j+\delta\beta_j\leq 0 \qquad (j=1,2,\ldots n)$$

Defining

$$\underline{\lambda} = \max_{\beta_j<0} \frac{-\alpha_j}{\beta_j} \text{ or } -\infty\text{, if all } \beta_j\geq 0$$

and

$$\overline{\lambda}= \min_{\beta_j>0} \frac{-\alpha_j}{\beta_j} \text{ or } +\infty\text{, if all } \beta_j\leq 0$$

The minimum solution will then be obtained for all such that

$$\underline{\lambda}<\lambda\leq\overline{\lambda}$$

If $\overline{\lambda} = +\infty$ then the solution is optimum over all admissible values of λ, $\delta\leq\lambda\leq\phi$. If, however, $\overline{\lambda}$ is finite, then, in particular $\overline{\lambda}=-\alpha_k/\beta_k$ for some particular $\beta_k>0$. If all the corresponding $x_{ik}\leq 0$, then no minimum (optimum) solution will exist for $\lambda>\overline{\lambda}$. If, however, at least one $x_{ik}>0$ then one can introduce a new vector P_k into the basis (by the simplex method technique). This new basis will result in a new range of optimality on $\bar{\lambda}$, namely:

$$\bar{\lambda}=\underline{\lambda}'\leq\lambda\leq\bar{\lambda}'$$

Thus, by successive iterations, one can proceed from one range of values of λ to the next, and completely cover all admissible value of λ, $\delta\leq\lambda\leq\phi$.

As noted by Gass (17), the various $\underline{\lambda}$ and $\bar{\lambda}$ that arise are called characteristic values of λ, while the corresponding optimum solutions are called characteristic solutions.

2. No finite optimum solution exists for $\lambda=\delta$. In attempting to determine an optimal (minimal) solution where $\lambda=\delta$, one has a column k, such that $\alpha_k+\delta\beta_k>0$. However, one cannot introduce a new vector into the basis because all $x_{ij}\leq0$.

a. If $\beta_k\geq0$, then no finite minimum solutions exist for any

b. If $\beta_k\leq0$, then $\alpha_k+\lambda\beta_k>0$ will hold for all

$$\lambda<\lambda_1' = -\frac{\alpha_k}{\beta_k}$$

Hence, no finite minimum solution will exist for $\delta\leq\lambda\leq\lambda_1'$.

If all $\alpha_j+\lambda_1'\beta_j\leq0$, then an optimum solution will exist for λ_1', and λ_1 can be determined by $\lambda_1=\min_{\beta_j>0}(-\alpha_j/\beta_j)$.

The characteristic solution holds for $\lambda_1'\leq\lambda\leq\lambda_1$, and one can then proceed as in the first case.

If all $\alpha_j+\lambda_1'\beta_j>0$ for at least one value of j, then a new basis can be obtained, and one can continue, finally obtaining a solution as in the first case, or the knowledge that there are not values of λ for which a finite minimum solution exists.

Summarizing (17) we have seen that:

1. By a modification of the general simplex procedure, it is possible to investigate systematically and solve the one parameter objective-function problem.
2. Given any finite minimum solution, we can determine a set of characteristic solutions and the associated characteristic values for all possible values of the parameter.
3. A solution is minimum over a closed interval of λ.
4. The set of λ for which minimum solutions exist is closed and connected.

The generalization of one-parameter linear programming problem to the case of the parameterization of the objective function with n parameters has been outlined by Gass and Saaty (16).

For the case of n=2, one seeks to minimize

$$\sum_{j=1}^{n} (c_j+\lambda_1 c'_j+\lambda_2 c''_j)x_j$$

and, generalizing on the method for one-parameter problem, one must determine the convex region in (λ_1,λ_2)-plane whose points satisfy

$$\alpha_j+\lambda_1\beta_j+\lambda_2\gamma_j\leq 0 \qquad (j=1,2,\ldots n)$$

Gass and Saaty consider two methods for so doing, namely, the double descriptive method (28) and the two-dimensional graph of inequalities, and illustrate their parametric programming procedure by the latter process.

The parametric-programming problem involving the right-hand-side coefficients can be stated mathematically (15) as follows:

Let $\sigma \leq \theta \leq \beta$. For each θ in this interval, find a vector $x = (x_1, x_2 \ldots x_n)$ such that

$$\min \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j = b_i + \theta b_i \qquad (i=1,2,\ldots m)$$

$$x_j \geq 0 \qquad (j=1,2,\ldots n)$$

This problem, however, can be considered in its dual formulation in which case one obtains a parametric objective function problem of the form considered before, which can then be solved by the procedure described therein.

In the general case Saaty (34) considers a more general parametric-programming problem in which all coefficients a_{ij}, c_j, and b_i are function of time. This problem can be cast in linear programming form in which the coefficients are functions of time. In fact, many linear programming problems occurring in application may be cast in this parametric form. For example, in the petroleum industry it has been found useful to parameterize the outputs as functions of time. In Leontieff models, this dependence of the coefficients on time is an essential part of the problem. Of special interest is the general case when inputs, the outputs, and the costs all vary with time. When the variation of the coefficients with time is known, it is then desired to obtain the solution as a function of time, avoiding repetitions for specific values.

This procedure requires solving sets of simultaneous general (not necessarily linear) inequalities in t, resulting from the conditions $z_j - c_j \leq 0$ and as Saaty observes is generally cumbersome except for problems involving the parameterization of the coefficients of only a few of the basis vector.

Another approach using the saddle point properties is the primal-dual method.

Primal: Max $c'x$ Dual: Min $y'b$

$$Ax \leq b \qquad A'y \geq c$$

$$x \geq 0 \qquad y \geq 0$$

If we define $\delta = (A+\delta A,\ b+\delta b,\ c+\delta c)$ by definition of a saddle point to

$$(\delta c' - y'\delta A)\ \delta x - \delta y\ (\delta b - \delta Ax) \geq 0$$

where the first part is termed the corrected change in probability and the second term adjusted capacity.

3.3. Other Measures

In an ordinary linear programming problem with a given set of statistical data, it is not known generally how reliable is an optimal basic solution and for that matter, any other basic feasible solution. One of the consequences of an ordinary small variations of the elements of the coefficient matrix, the elements of the resource vector or those of the vector of net prices in the objective function. Some methods of parametric programming have been developed and applied in situations where the parameters of the problem are known to change in a certain way.

An alternative form of sensitivity analysis is specified by considering solutions other than the optimal one and thereby initiating an approach to the theory of the second best.

This type of measurement divides the set of all feasible solutions into two subsets, the first containing all solutions except the basic feasible ones, the second containing only the basic feasible ones (i.e., the set of vertices of the convex polyhedron). For an ordinary and well-behaved linear programming problem, e.g., the primal maximization problem

$$\text{Max } z=c'x$$

subject to

$$Ax\leq b$$
$$x\geq 0 \qquad (3.3.1)$$

if the solution exists, then the second subset could be subdivided further into disjoint subsets, the first containing the maximum value and the objective function and the corresponding optimal solution and the second containing the rest. With a given objective function, it is possible to order the basic feasible solutions belonging to the latter subset in an increasing order according to objective function values. This permits us a working rule to define second best (or more precisely 'truncated') values of the objective function. For a detailed mathematical treatment of several theorems connected with truncated solutions of a stochastic linear programming problem the following reference (40) may be consulted.

Now let us denote the variations of the parameters (A,b,c) by an index set q or (A,b,c)q where q=1,2,...,Q runs over only admissible values. An admissible value is any value of the set (A,b,c)q which satisfies the conditions of an ordinary linear programming problem in the sense that the above described subsets containing the best, the second best, etc. values of the objective function are non-empty. For any fixed value of q and, hence, the set (A,b,c)q of a linear programming problem mentioned in (3.3.1), let the index $k=1,2,\ldots,K_q$ denote the set of basic feasible solutions. It is denoted by $F_q^{(k)}(z)$ the value of the objective function for a fixed q and a particular k. Now define

$$F_q^{(i)} = \underset{k}{\text{Max}} \; [F_q^{(k)}(z) \mid k=1,2,\ldots,K_q]$$

$$F_q^{(j)} = \underset{k}{\text{Max}} \; [F_q^{(k)}(z) \mid k=1,2,\ldots,K_q \text{ and } K \neq i] \qquad (3.3.2)$$

$$F_q^{(p)} = \underset{k}{\text{Max}} \; [F_q^{(k)}(z) \mid k=1,2,\ldots,K_q \text{ and } K \neq i,\ K \neq j]$$

It is assumed without loss of generality that our basic feasible solutions are so defined that $F_q^{(k)}$, $F_q^{(j)}$ and $F_q^{(p)}$ are strictly positive for all admissible q and that by construction

$$F_q^{(i)} > F_q^{(j)} > F_q^{(p)} > 0 \qquad (3.3.3)$$

since the weak inequalities

$$F_q^{(i)} \geq F_q^{(j)} \geq F_q^{(p)} \geq 0$$

can be easily reduced to strict inequalities by defining that each of the indices i, j and p may contain more than one point (i.e., more than one selection), provided they give rise to the same value of the objective function. For example, if there are three points (i.e., three basic feasible solutions) in the sequence $k=1,2,\ldots,K_q$ for a fixed sample q, which give rise to the identical maximum value $F_q^{(i)}$, then the superscript i contains these three points, so that in the definition of truncated maxima $F_q^{(j)}$, the condition $k \neq i$ has to be interpreted accordingly with suitable modifications. From now on it will be designated $F_q^{(i)}$ the regular maximum, i.e., truncated maximum of zero order (or, the best solution), $F_q^{(j)}$ as the truncated maximum of the first order (or, the second best solution) and $F_q^{(p)}$ as the truncated maximum of second order (or, the third best solution) and assume that the parameter variation are such that these three maximum values are generated for each admissible k, satisfying the conditions of an ordinary linear programming problem.

We note that there is an infinity of solutions between $F_q^{(i)}$ and $F_q^{(j)}$, i.e., the convex combination as moves from one to zero. However, we restrict ourselves only to the vertex points (i.e., basic feasible solution) for the derivation of a decision rule because the set of basic feasible solutions is finite and countable on the one hand and the activity vectors entering into the basic feasible solution are linearly independent, implying that the instrument variables included in the set of activity vectors are linearly independent.

Now we can consider the three truncated maxima $F_q^{(i)}$, $F_q^{(j)}$ and $F_q^{(p)}$ of order zero, one and two, respectively, as defined in (e.e.2) and the following two lemmas which characterize the truncated maxima (41).

Lemma 1. Let $Ax \leq b$ denote a set of constraints, which together with the nonnegativity requirement $x \geq 0$ define a closed and bounded convex set in the real domain. Then there exists another closed and bounded convex set, which is a proper subset of the convex set defined by $Ax \leq b$, $x \geq 0$ and which has as its extreme points those of $Ax \leq b$ except for one which is eliminated. In other words, any extreme point in $Ax \leq b$, $x \geq 0$ can be eliminated and all convex combinations of those remaining will define another bounded and closed region in the real domain.

Lemma 2. No two distinct sets of (m-1) of m bounding hyperplanes which intersect at $(x^o_1 \ldots, x^o_m)$ can both pass through a second extreme point.

And let us define over all admissible $q=1,\ldots,Q$ the expected values of $F_q^{(s)}$ by $E(F_q^{(s)})$ and the variance of $F_q^{(s)}$ by $Var(F_q^{(s)})$ where $s=i,j$ or p. From the relation (3.3.3) it readily follows that

$$E(F_q^{(i)}) > E(F_q^{(j)}) > E(F_q^{(p)})$$

if we find

$$Var(F_q^{(c)}) < Var(F_q^{(j)}) < Var(F_q^{(p)})$$

the optimal value $F_q^{(i)}$ is said to be stable. If, however, it turns out that

$$Var(F_q^{(j)}) < Var(F_q^{(i)})$$

and this difference in variance far outweights the difference in expected values it might be more reasonable to accept the second best solution $F_q^{(j)}$ which is more stable in terms of variance than the best one, $F_q^{(i)}$.

Sengupta (38) added two comments. First, the results in the theory of second best are applicable only for "wait-and-see" or passive situation. An active approach could be introduced, however, as is done in stochastic linear programming by introducing additional decision variables controlled by the decision-maker. Secondly, the above type of sensitivity indices, especially if modified to include the active approach, is very closely related to operational measures of sensitivity developed in physical sciences. As an example of the latter one may mention that the sensitivity of a circuit is usually expressed as the ratio of the difference between the maximum and minimum values of the output quantity to its mean value, i.e.,

$$\text{Sensitivity of a circuit} = \frac{\mu\ \max - \mu\ \min}{\text{mean}}$$

where μ = value of output quality.

The operations researcher (20) is often faced with devising models for operational systems. The systems usually contain both probabilistic and decision-making features, so that we should expect the resultant model to be quite complex and analytically intractable. This has indeed been the case for the majority of models that have been proposed. The exposition of dynamic programming by Richard Bellman (7) gave hope to those engaged in the analysis of complex systems, but this hope was diminished by the realization that more problems could be formulated by this technique than could be solved. Schemes that seemed quite reasonable often ran into computational difficulties that were not easily circumvented.

Howard (20) in his work provides an analytic structure for a decision making system that is at the same time both general enough and yet computationally feasible. It is based on the Markov process as a system model, and it uses an iterative technique similar to dynamic programming as its optimization method.

For a system operating under a fixed policy, a knowledge of the total expected reward of the process constitutes a complete understanding of the system. The most interesting cases arise when there are alternatives available for the operation of the system. In general, the problem is to find which set of alternatives of policy will yield the maximum total expected reward.

4. APPLICATION OF A MODEL OF PROBABILISTIC PROGRAMMING

Here we will try to show some application of the methods so far surveyed to the optimum organization of agricultural production in the Chincha Valley, Peru for illustrative purposes only. The applications are based on the data of a linear programming problem considered by Amorin in his thesis (1), where he pointed out that in the Chincha Valley uncertainty exists about the optimum combination of crops produced on any farm. This is reflected by the variety of different crops produced by the farmers of the area and also in the variations of yields rates and net return from the use of resources, especially capital and water.

His objectives of his study were as follows:

(a) to define the optimum combination of crops which maximizes the net income of small farms (i.e., a representative farm), considering the limitations of capital, land, labor and water in the Chincha Valley.

(b) to analyze capital restrictions at selected levels, since capital is one of the most critical limitations in Peru.

(c) to define the amount of land best suited for the resources of water, capital, and labor available on the farm.

The conclusions of his study were: (1) that the small farms of the Chincha Valley have an excess of family labor and (2) the main resource restrictions are water and capital.

He used linear programming techniques to solve his problem.

4.1. Definition of the Problem

We will use almost the same model as that of Amorin, which is designed to specify the plan which will give maximum income, considering the limitations of capital, water, land and labor; however the variations of incomes due to variation of prices of the products in the market are allowed in our case. Now for any given crop we have the relation

$$\text{Var (Income)} = \text{Var (price x yield)} = (\text{yield})^2 \times \text{Var (price)} \qquad (4.1.1)$$

if the price element only is random.

In the Chincha Valley eleven different crops were defined. These were crops that have been usually produced there with acceptable yields.

An estimate of net income and its standard deviation per hectarea by activities (crops) found in the area is shown in Table 1.

Table 1. Annual net income by crop and its standard deviation[1]

	Crop	Income	Standard deviation of income
x_1	Squash	9,023.00	94.9894
x_2	Peas	7,617.13	87.2761
x_3	Sweet potato	6,927.99	83.2345
x_4	Tomato	11,535.51	107.4034
x_5	Hybrid corn	6,030.39	77.6555
x_6	Beans with corn	10,182.54	100.9085
x_7	Alfalfa	9,644.11	98.2044
x_8	Cotton	4,669.37	68.3327
x_9	Lima beans	11,431.17	106.9166
x_{10}	Corn	9,295.15	96.4113
x_{11}	Yuca	13,018.25	114.0975

[1]The standard deviation is calculated according to the formula (4.1.1) where price variance were taken from (32), the yields from (12). However since the variance found there from was too great, we assumed a Poisson distribution, as an approximation according to which mean equal variance. Since this is an illustrative problem in risk programming, this assumption seems reasonable. With more data this assumption could be relaxed to allow more flexibility.

Table 2. Capital[1] and water[2] requirement by quarter and activity; for small farms in Chincha Valley

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
	Squash	Peas	Sweet Potato	Tomato	Hybrid corn	Bean with corn	Alfalfa	Cotton	Lima Beans	Corn	Yuca
CAPITAL											
1st quarter	3611	0	0	2141	0	0	0	6235	2640	0	1299
2nd quarter	3976	1163	1284	6707	0	860	1834	7138	3394	1212	1541
3rd quarter	790	3786	2585	0	2229	2718	2486	1395	3814	3642	184
4th quarter	2487	0	0	0	4904	0	3138	5133	4182	0	926
WATER											
1st quarter	4	0	0	2	0	0	2	4	2	0	2
2nd quarter	2	2	2	1	0	5	2	0	2	5	2
3rd quarter	5	5	2	0	5	25	2	3	2	4	2
4th quarter	4	0	0	0	2	4	2	5	2	0	2

[1]The values are expressed in sales.

[2]The values are expressed in irrigations for small farm, 288 cubic meter per irrigations.

Resource restrictions. In this model we have capital, land, labor and water as the most relevant restrictions. The difference between Amorin's model and the one presented here is that we considered the restriction quarterly. We have taken the median value of Amorin's monthly data of input-output coefficients and resource coefficients.

(a) Capital. The Banco de Fomento Agropecuario del Peru limits capital per month to 72 percent of the value of the land divided by 12 months. Assuming the value of land to be 50,000 soles per hectarea (1, pp. 20); then median of capital available per quarter will be:

$$\frac{50{,}000 \times \text{Land} \times .72}{12} \equiv 4{,}000 \times \text{Land}$$

where Land is equal to the number of hectarea; more details about capital availability in Peru can be found in (31). The requirement of capital by quarter for each activity is presented in Table 2.

(b) Water. The median quarterly restriction of water is shown in Table 2. The requirements of irrigations per crops are presented in Table 4.

(c) Labor. Amorin's model (1, pp. 24-25) did not consider hired labor; instead an average of six members per family was assumed. In our model a maximum of 480 hours of labor as a median per quarter was considered. Requirements of labor for each crop are presented in Table 3.

Table 3. Labor[1] requirement by quarter and activity for small farms in Chincha Valley

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
	Squash	Peas	Sweet potato	Tomato	Hybrid corn	Bean with corn	Alfalfa	Cotton	Lima beans	Corn	Yuca
1st quarter	52	0	0	58	0	0	10	28	26	0	2
2nd quarter	2	5	6	77	0	0	12	82	57	86	2
3rd quarter	0	78	6	0	70	52	12	35	57	0	42
4th quarter	42	0	0	0	38	104	52	31	42	0	34

[1]The values are expressed in hours.

Table 4. Resources available by quarter[1]

	Capital[2]	Water[3]	Labor[4]	Land
1st quarter	3,000 x L	8	480	L
2nd quarter	3,000 x L	10	480	L
3rd quarter	3,000 x L	12	480	L
4th quarter	3,000 x L	7	480	L

[1]Capital and land are parameters in function of the number of hectareas.

[2]The values are expressed in soles.

[3]The values are expressed in irrigations, 288 cubic meters per irrigation.

[4]It assumes 2 worker x 8 hours x 30 days.

(d) Land. The level of productivity of the soil is the actual average obtained in the Chincha Valley. Three different size groups of a typical farm (e.g. 4 hectareas, 8 hectareas and 16 hectareas) were used.

As it is seen the data have been primarily taken from Amorin's work. There are some variations however in the linear programming formulations, e.g. our problem limits the area in the valley allocated to yuca to one-fourth of the available land and not to one hectarea as in Amorin's programs.

For our linear programming formulations we have presented in the next section the following characteristics, e.g., the optimal solution, the second best and third best solution, the area of the triangle given by those three points and their respective distances.[1] The second best, third best and the area of the triangle provides an initial (non-probabilistic) measure of risk in the sense that they indicate the extent to which net total income may fall, in the event net prices vary, other restrictions being equal.

We can build a vector with the following components

$v = (z, AB, AC, \Delta)$

where

z is the objective function value of the program, AB is the euclidean distance between the optimal solution and the second best solution, AC the euclidean distance between the optimal solution and the third best solution, Δ is the area of the

[1]See Appendix for the calculation of triangle and distances.

triangle, with extreme points as optimal, second best and third best solutions.

Define vectors $v^{(1)}, v^{(2)}, \ldots, v^{(k)}$ defined above for k linear programming models

$$v^{(1)} = \begin{pmatrix} z \\ AB \\ AC \\ \Delta \end{pmatrix}^{(1)}, \quad v^{(2)} = \begin{pmatrix} z \\ AB \\ AC \\ \Delta \end{pmatrix}^{(2)}, \quad v^{(k)} = \begin{pmatrix} z \\ AB \\ AC \\ \Delta \end{pmatrix}^{(k)}$$

then a partial ordering (introduced by the decision maker) defined over sets $v^{(k)}$ makes subjective comparison between two problem comparable (this is comparable to the concept of efficiency in the sense of Koopmans for linear programming problems with a vector objective function).

4.2. Static Cases to be Studied: I, II, III, IV, V

Here we point out the most important cases, we have studied using linear programming techniques; it is static in the sense that we have taken only one observation, of the c_i, a_{ij}, and b_j coefficients of the general linear programming problem.

Case I. Optimum farm plan assuming 4 hectareas of farm and 12,000 soles of monthly average restriction of capital and the restrictions of water and labor as indicated in Table 4; no restriction on land to be allocated to yuca.

The most binding restriction is land in the 4th quarter, we have two activities, tomato and yuca. The value of the objective function is 51,331.367 soles. The results in detail are indicated in Table 5.

Case II. Optimum farm plan assuming 4 hectareas of farm and 12,000 soles of monthly average restriction of capital and the restriction of water and labor as indicated in Table 4; yuca has an upper brand of one hectarea.

The most binding restriction is also land in the 4th quarter, we have four activities; tomato, lima beans, corn, and yuca. The value of the objective function is 46,801.289 soles.

The results in detail are indicated in Table 6.

Case III. Optimum farm plan assuming 8 hectareas of farm and 24,000 soles of monthly average restriction of capital and the restriction of water and labor as indicated in Table 4; yuca has an upper bound of 2 hectareas.

The most binding restriction is water in the second quarter, we have four activities; sweet potato, tomato, hybrid corn, and yuca. The value of the objective function is 70,494.023 soles.

The results in detail are indicated in Table 7.

Table 5. Optimum farm plan as indicated in case I

Activity code	Activity description	Optimal solution (A) level of activity (hectareas)	Second best solution (B) level of activity (hectareas)	Third best solution (C) level of activity (hectareas)
x_4	tomato	0.50001	1.12969	0.50001
x_9	lima beans	-	-	1.75550
x_{11}	yuca	3.49997	2.87030	1.74448
Value of the program (z)		51331.367	50397.695	48545.23

Area of the triangle ABC = .95729864
Distance AB = .89049447
Distance AC = 2.48264410
Distance BC = 2.17847250

Table 6. Optimum farm plan as indicated in case II

Activity code	Activity description	Optimal solution (A) level of activity (hectareas)	Second best solution (b) Level of activity (hectareas)	Third best solution (C) Level of activity (hectareas)
x_4	Tomato	.24895	-	-
x_9	Lima beans	2.49999	2.49999	2.33332
x_{10}	Corn	0.25105	0.50000	0.66667
x_{11}	Yuca	1.00000	1.00000	1.0000
Value of the program (z)		46801.289	46243.516	45887.500

Area of the triangle ABC = 0.035933215
Distance AB = .35206836
Distance AC = .51234245
Distance BC = .23570681

Table 7. Optimum farm plan as indicated in case III

Activity code	Activity description	Optimal solution (A) level of activity (hectares)	Second best solution (B) Level of activity (hectareas)	Third best solution (C) Level of activity (hectareas)
x_3	Sweet potato	2.42493	2.00000	2.00000
x_4	Tomato	2.84986	2.00000	2.00000
x_5	Hybrid corn	0.96997	0.80000	-
x_{11}	Yuca	1.15014	2.00000	2.00000
Value of the program (z)		70494.023	67785.625	62961.602

Area of the triangle ABC = .50991368
Distance AB =1.28606990
Distance AC =1.60185150
Distance BC = .79999995

Table 8. Optimum farm plan as indicated in case IV

Activity code	Activity description	Optimal solution(A) level of activity	Second best solution(B) level of activity	Third best solution (C) level of activity
x_3	Sweet potato	3.0	3.0	1.25
x_4	Tomato	4.0	4.0	0.50000
x_5	Hybrid corn	1.2	-	-
x_{11}	Yuca	-	-	3.5000
Value of the program (z)		74159.258	66923.24	59990.352

Area of the triangle ABC = 3.14999580
Distance AB = 1.19999890
Distance AC = 5.38539510
Distance BC = 5.25000000

Table 9. Optimum farm plan as indicated in case V

Activity code	Activity description	Optimal solution (A) level of activity (hectareas)	Second best solution (B) level of activity (hectareas)	Third best solution (C) level of activity (hectareas)
x_3	Sweet potato	2.42691	1.56249	-
x_4	Tomato	2.85381	1.12500	2.16666
x_5	Hybrid corn	0.97076	0.62500	1.66666
x_7	Alfalfa	0.14618	1.87500	0.83333
x_{11}	Yuca	1.00000	1.00000	1.00000
Value of the program (z)		70013.359	58670.664	56098.477

Area of the triangle ABC = 2.835961
Distance AB = 2.6161737
Distance AC = 2.7052746
Distance BC = 2.3867455

Table 10. Simplex multipliers

	Restrictions		Case I	Case II	Case III	Case IV	Case V
1	Capital	1st quarter	-	-	-	-	-
2		2nd quarter	-	.4070	.5487	-	1.0999
3		3rd quarter	-	-	-	-	-
4		4th quarter	-	-	-	-	-
5	Water	1st quarter	-	-	297.5049	463.8972	130.3496
6		2nd quarter	-	-	190.5301	225.7562	155.1420
7		3rd quarter	-	-	120.6000	120.6000	120.5999
8		4th quarter	74.1376	62.3200	-	-	-
9	Labor	1st quarter	-	-	-	-	-
10		2nd quarter	-	-	-	-	-
11		3rd quarter	-	-	-	-	-
12		4th quarter	-	-	-	-	-
13	Land	1st quarter	-	-	-	-	-
14		2nd quarter	-	-	-	-	-
15		3rd quarter	-	-	-	-	-
16		4th quarter	1153.553	880.1018	-	-	-

Case IV. Optimum farm plan assuming 16 hectareas of farm and 48,000 soles of monthly average restriction of capital and 48,000 soles of monthly average restriction of capital and the restriction of water and labor as indicated in Table 4; yuca has an upper bound of 4 hectareas.

The most binding restriction is water in the second quarter, we have three activities; sweet potato, tomato, and hybrid corn. The value of the objective function is 74,159.258 soles.

The results in detail are indicated in Table 8.

Case V. Optimum farm plan assuming 8 hectareas and 24,000 soles of monthly average restriction of capital and the restriction of water and labor as indicated in Table 4; yuca has an upper bound of 1 hectarea.

The most binding restriction is water in the second quarter, we have five activities; sweet potato, tomato, hybrid corn, alfalfa, and yuca. The value of the objective function is 70013.359 soles. We will use this case as the basic starting solution for the next case.

In Table 10 we show the simplex multiplier associated at the non-structural variables for the 16 restrictions for every case studied in this section.

4.3. A Chance-Constrained Programming, Case VI

Here we have a chance-constrained problem:

$$\text{Max } f \tag{4.3.1}$$

subject to

$$\text{Prob } (c'x \leq f) = \alpha \tag{4.3.2}$$

$$\sum_j a_{ij}x_j \leq b_i \tag{4.3.3}$$

$$x_j \geq 0$$

$(i=1,2,\ldots 16 \quad j=1,2,\ldots 11)$

Case VI is an optimum farm plan assuming 8 hectareas with an upper bound of 1 hectarea of yuca. The restriction indicated by (4.3.3) are the same as in case V: The c vector are the net incomes given in Table 1 and the standard deviation are also given in Table 1.

As we have seen in section 2.3 we can arrive at the following equivalent quadratic programming model.

$$\text{Max } f_{11} = \bar{c}'x - \frac{q}{2R}\ x'Vx \tag{4.3.4}$$

subject to

$$Ax<b$$
$$x \geq 0$$

where $q = I^{-1}(\alpha) = 2.33$ for $\alpha = .01$ if we assume that the function $I(\alpha)$ is a normal distributed function[1], and R is

[1]It might approximated the Poisson distribution to a normal distribution without too much error.

evaluated by the following iteration procedure (21, pp. 192):

Step 1. Start by solving the linear programming problem, i.e. make the quadratic part equal to zero in (4.3.4), obtain an initial value R, as R_o and store it in R.

$$R_o = \sqrt{x_o' V x_o} \rightarrow R$$

Step 2. Using this R, solve the equivalent quadratic problem (4.3.4) if the new value of f_{11} does not differ too much from the previous one, stop the problem. We have used the following stopping rule:

$$\frac{|R_{n-1} - R_n|}{R_n} \leq \varepsilon$$

where ε=.003 in our case.

For solving the quadratic programming problem we used the ZORILLA program (42) using the IBM 360 model 50 of the I.S.U. Computer Center, the average time was 1.84 minutes by iteration. In Table 10 we show the iterations to solve it.

In Table 12 we indicate the Lagrange multiplier for the 3 iterations. The most binding restriction is the yuca upper bound of 1 hectarea. The value of objective function is 69533.455 soles and we have six activities; sweet potato, tomato, hybrid corn, alfalfa, lima beans, and yuca.

Table 11. Katooka's iteration procedure

Iteration	$\frac{\lvert R_{n-1}-R_n \rvert}{R_n}$	$R=\sqrt{\hat{X}_o' V \hat{X}_o}$	Value of the program	Sweet potato x_3	Tomato x_4	Hybrid corn x_5	Alfalfa x_7	Lima bean x_9	Yuca x_{11}
L.P.	-	-	70013.359	2.42691	2.85381	.97076	.14618	-	1.0000
1st	-	391.9881	69533.427	2.413556	2.8227112	.9654223	.0936442	.07924418	1.0000
2nd	0.005	389.8882	69533.427	2.413556	2.8227116	.9654223	.0936442	.07924418	1.0000
3rd	0.002	389.0313	69533.455	2.413569	2.8271392	.9654278	.0936934	.0791629	1.0000

Table 12. Lagrange multiplier associated with the Q.P.

Code	Restriction	1st iteration	2nd iteration	3rd iteration
C13	Capital 2nd quarter	.11455467	.1145546	.11455468
C16	Water 1st quarter	109.01215	109.01215	109.01202
C17	Water 2nd quarter	147.96718	147.96178	147.96715
C18	Water 3rd quarter	119.91306	119.91306	119.91306
C28	Yuca ≤ 1 hectarea	363.83066	363.83066	363.83097

5. SUMMARY AND CONCLUSION

1. A brief survey of the main theoretical results on risk programming is presented in the first three chapters.

2. For our empirical illustration we have used the data of an optimal farm in Chincha Valley. These data were previously analyzed by Amorin (1, pp. 27-49) for his studies on linear programming.

In an ordinary linear programming problem with a given set of statistical data, it is not known generally how reliable is the optimal basic solution. In our five cases of linear programming we have only had one sample observation. It could be possible to indicate a more general method of reliability analysis for testing the sensitivity of the optimal basic solution and other basic solution, in terms of expectation and variance when more sample observations are available.

The first, second, and third best solutions are estimated for our linear programming models assuming the vectors of net income, resources and input-output matrix to be constant.

In every case studied the three alternatives are given to the farmer, he could decide what level of activities would satisfy his satisfying approach, in the event the optimal (i.e. the first best) solution is considered more risky.

In every case the triangle area gives us a measure of risk when we change from one extreme point to another. If we could have more sample values of the elements c_j (the price of the crops) it would be possible to see clearly our odds in making an optimal production planning. In this sense the second and third best solution specify suboptimal solutions.

We also give the simplex multipliers associated with every optimal basic solution (first best). The simplex multiplier $\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{16}$ can be used to compute the relative cost factor $\bar{c}_j$ from the corresponding column of the original system by the formula

$$\bar{c}_j = c_j - (\Pi_1 a_{1j} + \Pi_2 a_{2j} + \ldots\ldots + \Pi_{16} a_{16j})$$

3. A chance-constrained version of the linear programming model is then considered to see the sensitivity of the solutions and an equivalent quadratic program is formulated. Although the value of the program of the equivalent quadratic problem and level of activities do not differ significantly from the linear programming problem; we see that the linear programming solutions satisfy the chance constraints to a marked degree. However, if the tolerance measure (α) is varied, or the sampling distributions of the net unit returns are different from Poisson, results different from the above are quite expected.

4. A few concluding remarks may be added about the limitations and possible generalizations of our empirical approach. First, the variation of parameters (e.g. net prices) in our model is not specifically estimated for lack of comparable and homogeneous data. However, given more time and more data,

these parameters could be statistically estimated with more precision and then the effects of alternative distributions like normal or chi-square, to on the optimal decision rule could be compared and evaluated. Second, the second best, and third best solution with the area of the triangle could be used as a probabilistic measure for analyzing the sensitivity of any linear programming problem, provided statistical distribution of the parameter is known or estimated. Third, it can be argued that different levels of tolerance measure (i.e. different α) could be associated with the objective function and with different restrictions to see the "implicit cost" of flexibility in the sense of infeasibility. A scope for comparing safety first method with the chance constrained model exists for any feasible linear program and this seems to be a fruitful line of future research.

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8. APPENDIX

The area of a triangle with sides a,b,c is

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where

$$s = \frac{1}{2}(a+b+c)$$

In our problems we have points defined by n coordinates so we assume euclidean distances, for example if point A is defined by $(a_1,a_2,\ldots,a_n)$ and point B is defined by $(b_1,b_2,\ldots,b_n)$

$$D = \sqrt{(a_1-b_1)^2+(a_2-b_2)^2+\ldots+(a_n-b_n)^2}$$

or

$$D = \sum_{i=1}^{n} (a_i-b_i)^2$$

The following is a FORTRAN IV program to make possible the calculation.

Table 13. Fortran program to calculate the area of triangle

```
C     PRØGRAM TØ CALCULATE THE AREA ØF A TRIANGLE, GIVEN THE N-CØØRDI
C     NATES ØF THE EXTREME PØINTS A,B,C
      DIMENSIØN A(50), B(50), C(50)
C     READ A CARD WITH THE NUMBER ØF CØØRDINATES
      READ (1,2) NPØINT
    2 FØRMAT (I3)
C     SET ALL THE PØINTS TØ ZERØ
    1 DØ 3  I=1, NPØINT
      A(I) = 0.
      B(I) = 0.
    3 C(I) = 0.
      D1 = 0.
      D2 = 0.
      D3 = 0.
C     READ VALUES ØF THE CØØRDINATES A,B,C INDIFICATIØN IND  A=1  B=2
C     C = 3, IND = 4  ENDS THE SET ØF VALUES
    4 READ (1,5)   IND, K, X
    5 FØRMAT (I2, 2X, I3, 2X, F12.6)
    6 A(K) = X
      GØ TØ 4
    7 B(K) = X
      GØ TØ 4
```

Table 13 (Continued)

```
  8 C(K) = X
    GØ TØ 4
  9 DØ 10  I=1,  NPØINT
    D1=D1 + (A(I)-B(I))**2
    D2=D2 + (A(I)-C(I))**2
 10 D3=D3 + (B(I)-C(I))**2
    DE1=SQRT(D1)
    DE2=SQRT(D2)
    DE3=SQRT(D3)
    P = (DE1 + DE2 + DE3)/2.
    AREA = SQRT(P*(P-DE1)*(P-DE2)*(P-DE3))
C   TITLE WRITING
    WRITE(3,11)  NPØINT
 11 FØRMAT ('1',30X,'PRØGRAM TØ CALCULATE THE AREA ØF A TRIANGLE IN',2
   1X,I3,' CØØRDINATES',////,'0',30X,'I',20X,'A(I)',16X,'B(I)',27X,'C
   2(I)',////)
    DØ 12 K=1,  NPØINT
 12 WRITE (3,13) K, A(K), B(K), C(K)
 13 FØRMAT ('0',27X,I3,13X,E14.8,7X,E14.8,15X,E14.8)
    WRITE (3,14) DE1, DE2, DE3, AREA
 14 FØRMAT ('0',///,15X,'DIST A-B',2X,E15.8,10X,'DIST A-C',2X,E15.8,1
   10X, 'DIST B-C', 2X, E15.8,///'0'.35X,'AREA ØF TRIANGLE EQUAL TØ
   2, E15.8,//,'0',30X, '**************        ***********')
```

Table 13 (Continued)

```
C     A NEW TRIANGLE THEN CHECK
      READ (1,15)  NPØINT
   15 FØRMAT (I3)
      IF (NPØINT) 16,16,1
   16 STØP 0007
      END
```